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Determination of all the Characteristic Subgroups of any Abelian Group.

By G. A. Miller.

§1.—*Statement of the Problem and of the Principal Results.*

A subgroup which corresponds to itself in every possible simple isomorphism of the group (G) with itself, is called a characteristic subgroup.* The term was first used by Frobenius in Berliner Sitzungsberichte, 1895, p. 183. Burnside observed that a group which has no characteristic subgroup is either simple or the direct product of simply isomorphic simple groups.† The present paper is devoted to a determination of all the possible characteristic subgroups of G whenever G is abelian. In what follows it will be assumed that all the groups under consideration are abelian.

In every possible simple isomorphism of G with itself, every Sylow subgroup‡ must correspond to itself. Hence, it follows that the necessary and sufficient condition that a subgroup of G is characteristic is that each of the Sylow subgroups of this subgroup is characteristic in a Sylow subgroup of G. If we include the identity and G itself in the term characteristic subgroup, it follows that the number of the characteristic subgroups of G is the product of the numbers of the characteristic subgroups in all the Sylow subgroups of G. That is, the characteristic subgroups of G are the direct products of characteristic subgroups of the Sylow subgroups of G. Hence we may confine our attention to the case where the order of G is p^m, p being any prime number. This will be done in what follows unless the contrary is explicitly stated.

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* A characteristic subgroup may also be defined as a subgroup which is transformed into itself by all the operators in the group of isomorphisms of G.

† Burnside, Theory of Groups of Finite Order. 1897, p. 232.

‡ If the order of a group is divisible by p^m but not by p^{m+1}, a subgroup of order p^m, is called a Sylow subgroup. The term was first used in Bulletin of the American Mathematical Society, Vol. 9 (1903), p. 543.

The principal results are as follows:

When p is odd, two characteristic subgroups cannot be of the same type, and the number of characteristic subgroups in G is equal to the number of its sets of operators such that each set is composed of all the operators which are conjugate under the group of isomorphisms (I) of G. All the possible characteristic subgroups (besides the identity*) have a certain characteristic subgroup (C_1) in common. This is called the *fundamental characteristic subgroup* of G and is the only one in which all the operators besides the identity are conjugate under I. If a characteristic subgroup includes an operator (s) which could be used as an independent generator of G, then it includes all the operators of G whose orders divide the order of s. In particular, if a characteristic subgroup includes an operator of highest order, it must be G itself. If a characteristic subgroup includes k invariants, which are equal to or greater than p^δ, it must include all the characteristic subgroups which do not have more than k invariants, provided none of these invariants exceeds p^δ. The total number of the characteristic subgroups of G can be directly obtained by means of a simple formula.

§2.—*The Characteristic Subgroups which are Generated by Operators of Order p.*

Although the results of this section are special cases of those which will be developed in the following sections, yet it seems desirable to give them separately on account of their simplicity and their fundamental importance in the theory of characteristic subgroups. The treatment of the following sections can be presented in a somewhat briefer form in view of these special developments.

Suppose that β_1 invariants of G are equal to p^{α_1}, β_2 are equal to p^{α_2}, ..., β_λ are equal to p^{α_λ}; where $\alpha_1 > \alpha_2 > \dots > \alpha_\lambda > 0$. Hence $\alpha_1\beta_1 + \alpha_2\beta_2 + \dots + \alpha_\lambda\beta_\lambda = m$. Representing a set of independent generators corresponding to these invariants, in order, by $s_1, s_2, \ldots, s_{\beta_1+\beta_2+\ldots+\beta_\lambda}$, the group G may be regarded as the direct product of the subgroups

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* In what follows, the identity will not be included in the term characteristic subgroup, but G will be regarded as a characteristic subgroup of itself.

$$H_1, H_2, \ldots, H_\lambda,$$

which are respectively generated* by the independent generators of the same order, as follows:

$$H_1 \equiv \{s_1, s_2, \ldots, s_{\beta_1}\}, \quad H_2 \equiv \{s_{\beta_1+1}, s_{\beta_1+2}, \ldots, s_{\beta_1+\beta_2}\} \ldots$$
$$H_\lambda \equiv \{s_{\beta_1+\beta_2+\ldots+\beta_{\lambda-1}+1}, s_{\beta_1+\beta_2+\ldots+\beta_{\lambda-1}+2}, \ldots, s_{\beta_1+\beta_2+\ldots+\beta_\lambda}\}.$$

All the operators of the same order in H_{λ_1}, $\lambda_1 \leqq \lambda$, are conjugate under its group of isomorphisms (I_{λ_1}), since each of these operators is the same power of operators which could be selected separately as independent generators of H_{λ_1}. These operators are also conjugate under I since I includes the direct product of $I_1, I_2, \ldots, I_\lambda$.† If we multiply any operator of order p^{α_1} in H_1 by any operator in $\{H_2, H_3, \ldots, H_\lambda\}$ and raise the product to the p^{α_1-1} power, the constituent from $\{H_2, H_3, \ldots, H_\lambda\}$ will reduce to the identity. That is, H_1 includes all the operators of order p in G which are found in cyclic subgroups of order p^{α_1}, and it contains no other operators of order p. In other words, the identity and the operators of order p in H_1 constitute a characteristic subgroup (C_1') of G. It will soon be proved that $C_1' \equiv C_1$.

The group $\{H_1, H_2\}$ contains all the operators of order p in G which are contained in cyclic subgroups of order p^{α_2}, and all of its operators of order p have this property. Hence these operators, together with the identity, constitute a second characteristic subgroup (C_2) of G. All the operators of C_2, which are not also in C_1, are conjugate under I, since the independent generators of H_2 might have been so selected that the p^{α_2-1} power of one of them would give any given one of these operators. Since a characteristic subgroup may be defined as a subgroup which includes all the conjugates under I of its own operators, and since a group may always be generated by the operators which do not belong to any given subgroup, it follows that every characteristic subgroup which involves an operator of order p from C_2 must include C_1'.

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* This symbol denotes the group generated by all the operators inclosed.

† Transactions of the American Mathematical Society, Vol. 1 (1900), p. 396.

In an exactly similar manner we observe that all the operators of order p in $\{H_1, H_2, \ldots, H_\epsilon\}$, $1 < \varepsilon \overset{=}{<} \lambda$, are composed of the operators of order p in the cyclic subgroups of G whose order is p^{α_ϵ}. Hence, these operators, together with the identity, constitute a characteristic subgroup (C_ϵ), which includes C_1', $C_2 \ldots, C_{\epsilon-1}$. All the operators of C_ϵ, which are not also in $C_{\epsilon-1}$, are conjugate under I because each of them is the p^{α_2-1} power of an operator which could have been selected as an independent generator of H_ϵ. Hence, there are just λ characteristic subgroups of G which are generated by operators of order p. The number of invariants of these subgroups are respectively β_1, $\beta_1 + \beta_2$, $\beta_1 + \beta_2 + \beta_3, \ldots, \beta_1 + \beta_2 + \ldots + \beta_\lambda$. Since every characteristic subgroup must involve operators of order p, and since C_ϵ includes $C_{\epsilon-1}$, it follows that C_1' *must be contained in every possible characteristic subgroup of* G. That is, $C_1' \equiv C_1$.

§3.—*The Conjugates under I of any Operator of G.*

Let s represent any operator of order p^δ in H_γ $(\delta \overset{=}{<} \alpha_\gamma, \gamma \overset{=}{<} \lambda)$. It has been observed that all the operators of order p^δ in H_γ are conjugate under I. We proceed to find all the other conjugates of s under I. Consider all the products obtained by multiplying all the operators of order p^δ in H_γ into all the operators whose orders do not exceed p^δ in $\{H_1, H_2, \ldots, H_{\gamma-1}\}$. As all these products are the $p^{\delta_\gamma-1}$ power of operators which could be chosen separately as independent generators of G, they must be conjugate under I. Since G may be regarded as the direct product of the three groups $\{H_1, H_2, \ldots, H_{\gamma-1}\}$, H_γ, $\{H_{\gamma+1}, H_{\gamma+2}, \ldots, H_\lambda\}$, it remains only to find the operators in the last one of these groups which are such that the products obtained by multiplying them into an operator of order p^δ in H_γ are conjugate with s. This follows directly from the fact that all the conjugates of s under I must be such that their $p^{\delta-1}$ power is in C_γ but not in $C_{\gamma-1}$.

Since s is the $p^{\alpha_\gamma-\delta}$ power of an operator which may be used as an independent generator of G, each of its conjugates must have the same property. Hence, the factor from $H_{\gamma+\gamma_1}$ $(\gamma_1 = 1, 2, \ldots, \lambda - \gamma)$ may be any operator whose order does not exceed $p^{\alpha_{\gamma+\gamma_1}-\alpha_\gamma+\delta}$ $(\alpha_{\gamma+\gamma_1} - \alpha_\gamma + \delta > 0)$, but no other operator of $H_{\gamma+\gamma_1}$ can be multiplied into s without affecting the system of conjugates to which s belongs, since such a product would not be the $p^{\alpha_\gamma-\delta}$ power of an operator which could be used as an independent generator* of G. Hence, *all the conjugates of s under I can be obtained by multiplying all the operators of order p^δ in H_γ by the group generated by all the operators whose orders divide p^δ in $H_1, H_2, \ldots\ldots, H_{\gamma-1}$, together with the operators whose orders divide $p^{\alpha_{\gamma+\gamma_1}-\alpha_\gamma+\delta}$ in $H_{\gamma+\gamma_1}$*. The number of these conjugates is, therefore, $p^l\,(p^{\beta_\gamma}-1)$, where

$$l=\delta\,(\beta_1+\beta_2+\ldots\ldots+\beta_\gamma)-\beta_\gamma+\sum_{\gamma_1=1}^{\gamma_1=\lambda-\gamma}\beta_{\gamma+\gamma_1}(\alpha_{\gamma+\gamma_1}-\alpha_\gamma+\delta),$$

$$\alpha_{\gamma+\gamma_1}-\alpha_\gamma+\delta>0.$$

That is, the last summation is to extend only to such values of γ_1 as will make $\alpha_{\gamma+\gamma_1}-\alpha_\gamma+\delta>0$. In other words, only the positive terms of the series are to be combined. In particular, when $\delta=\alpha_\gamma$, we arrive at the formula of the last foot-note.

The preceding method can be employed to determine all the conjugates under I of operators which are powers of possible independent generators. All the operators of orders p and p^{α_1} in G possess this property. The necessary and sufficient condition that all the operators of G are powers of possible independent generators is that either $\lambda=1$ or that both of the following conditions are satisfied: $\lambda=2$, $\alpha_1-\alpha_2=1$. We proceed to consider the number of conjugates of an operator (t) of G when t is not some power of a possible independent generator of G.

The constituents of t which belong to $H_1, H_2, \ldots\ldots, H_\lambda$ may be denoted respectively by $t_1, t_2, \ldots\ldots, t_\lambda$ and their orders by $p^{\alpha'_1}, p^{\alpha'_2}, \ldots\ldots, p^{\alpha'_\lambda}$. If the order of t is p^δ, at least one of the numbers $\alpha'_1, \alpha'_2, \ldots\ldots, \alpha'_\lambda$ must be equal to δ, while each of the others is equal to or less than δ. Let γ be the largest subscript of a constituent of order p^δ. From the fact that the constituents of t are separately powers of possible independent generators of G, it follows that there

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* Any operator of order p^δ in any abelian group whatever can be used as an independent generator, provided its $p^{\delta-1}$ power is not included in some cyclic subgroup of order $p^{\delta+1}$. Hence, all the operators of order p^{a_γ} which can be used as independent generators of G are the products of operators of highest order in H_γ into the group generated by all the operators of $\{H_{\gamma+1}, H_{\gamma+2}, \ldots\ldots, H_\lambda\}$ and the operators of $\{H_1, H_2, \ldots\ldots, H_{\gamma-1}\}$ whose orders divide p^{a_γ}. The number of these operators is

$$\{p^{\beta_\gamma a_\gamma}-p^{\beta_\gamma(a_\gamma-1)}\}\,p^{a_\gamma(\beta_1+\beta_2+\ldots\ldots+\beta_{\gamma-1})+\beta_{\gamma+1}a_{\gamma+1}+\beta_{\gamma+2}a_{\gamma+2}+\ldots\ldots+\beta_\lambda a_\lambda}.$$

is at least one conjugate of t such that both of the following conditions are satisfied:

$$\alpha'_\rho \geqq \alpha'_{\rho+1}, \quad \alpha_\rho - \alpha'_\rho \geqq \alpha_{\rho+1} - \alpha'_{\rho+1}, \qquad \rho = 1, 2, \ldots, \lambda - 1.$$

The necessary and sufficient condition that t is not a power of some operator which could be used as an independent generator of G is that $\alpha'_{\gamma+\gamma_1} > \alpha_{\gamma+\gamma_1} - \alpha_\gamma + \delta$ for at least one value of $\gamma_1 = 1, 2, \ldots, \lambda - \gamma$. We shall suppose that t satisfies all of these conditions. In other words, we shall suppose that t was so selected that it is not a power of some operator which could be used as an independent generator of G, and also that the order of each of its constituents is at least as large as the order of the corresponding constituent in any one of its conjugates.

All the conjugates of t can be obtained from the fact that all the conjugates of its constituents are known. From the preceding paragraphs, it can be seen that the following method gives all these conjugates: Construct the group formed by all the operators whose orders divide $p^{\alpha'_\rho}$ in the constituents in which at least one of the following two conditions is satisfied:

$$\alpha'_\rho = \alpha'_{\rho+1}, \quad \alpha_{\rho-1} - \alpha'_{\rho-1} = \alpha_\rho - \alpha'_\rho,$$

and multiply each of the operators of this group into all the operators obtained by multiplying together all the operators of order $p^{\alpha'_\rho}$ in the constituents for which both of the following conditions are satisfied:

$$\alpha'_\rho > \alpha'_{\rho+1}, \quad \alpha_{\rho-1} - \alpha'_{\rho-1} > \alpha_\rho - \alpha'_\rho.$$

If t is regarded as any operator of G, all the conjugates of t must involve at least one constituent whose order is the same in all of the conjugates. If it involves only one such constituent, t is a power of some operator which could be used as an independent generator of G; if it involves more than one such constituent, then t cannot be a power of a possible independent generator. It is also clear that the number of conjugates of an operator which is not a power of a possible independent generator is always larger than the number of conjugates of some operator of the same order which is a power of a possible independent generator. For the present purpose it is only necessary to bear in mind that the given method leads to all the conjugates of t under I.

The group generated by all the conjugates of t is evidently the direct product of all the groups formed by all the operators whose orders divide $p^{\alpha'_\rho}$ in H_ρ ($\rho = 1, 2, \ldots, \lambda$) whenever $p > 2$.* When $p = 2$, this result remains true only if there is no more than one cyclic H_ρ for which both of the conditions

$$\alpha'_\rho > \alpha'_{\rho+1}, \quad \alpha_{\rho-1} - \alpha'_{\rho-1} > \alpha_\rho - \alpha'_\rho$$

are satisfied. If there are $k > 1$ such constituents when $p = 2$, the group generated by the conjugates of t is the direct product of the group obtained by dimidiating† the cyclic subgroups of order $p^{\alpha'_\rho}$ of all these k constituents, and the subgroups which are composed of all the operators whose orders divide $p^{\alpha'_\rho}$ in the other H_ρ's. The order of the group generated by the conjugates of t is therefore

$$p^{\alpha'_1\beta_1 + \alpha'_2\beta_2 + \ldots + \alpha'_\lambda\beta_\lambda}$$

except when both of the conditions $p = 2$, $k > 1$ are satisfied. In this case the order is this number divided by 2^{k-1}.

§4.—*Determination of all the Characteristic Subgroups of G.*

The simplest case presents itself when all the invariants of G are equal; that is, when $G \equiv H_1$. It was observed above that all the operators of the same order are conjugate under the I of such a G, and hence there are just α_1 characteristic subgroups. These are composed respectively of all the operators of G whose orders divide p^β, $\beta = 1, 2, \ldots, \alpha_1$. Each of them contains just β_1 equal invariants and is of order $p^{\beta\beta_1}$. It is evident that all the operators whose orders divide p^β must always constitute a characteristic subgroup of G and that G contains additional characteristic subgroups whenever it has at least two unequal invariants.*

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* This result follows directly from the facts that the conjugates of t must generate a group which can be obtained by establishing some isomorphism between the subgroups of the H_ρ's, which are composed of the operators whose orders divide $p^{\alpha'_\rho}$, and that all the operators of such a subgroup must correspond to a given operator whenever all the operators of the highest order have this property except when both H_ρ is cyclic and $p = 2$.

† Cayley, Quarterly Journal of Mathematics, Vol. 25 (1890), p. 71.

We proceed to determine all the characteristic subgroups of G which involve C_ρ, but do not include $C_{\rho+1}$, $0 < \rho \leqq \lambda$. Let C be such a characteristic subgroup, and suppose that t is an operator of C which has been so selected that each of its constituents $t_1, t_2, \ldots, t_\lambda$ from $H_1, H_2, \ldots, H_\lambda$ respectively is of the largest possible order. As in the preceding section, we may suppose that the orders of these constituents (which exceed unity) are $p^{\alpha_1'}, p^{\alpha_2'}, \ldots, p^{\alpha_\rho'}$ respectively. Since C may be defined as any subgroup of G which includes all the conjugates under I of each one of its operators, it is clearly generated by the conjugates of t and its order is

$$p^{\alpha_1'\beta_1 + \alpha_2'\beta_2 + \ldots + \alpha_\rho'\beta_\rho}$$

whenever $p > 2$. That is, *when* $p > 2$, *each of the characteristic subgroups of* G *is generated by one and only one complete set of conjugate operators under* I, *and the independent generators of such a subgroup may be so selected that they are powers of the independent generators of* G.

All the C's in question can be obtained as follows:

The ρ invariants may all be equal to p. Then the first invariant may be successively multiplied by p without affecting the others until it is equal to $p^{\alpha_1 - \alpha_2 + 1}$. After this, the second invariant may be made equal to p^2 and the first invariant may be successively increased from p^2 to $p^{\alpha_1 - \alpha_2 + 2}$, etc. To each such set of invariants there corresponds a C. Hence, the number of the characteristic subgroups of G which contain C_ρ without also containing $C_{\rho+1}$ is

$$(\alpha_1 - \alpha_2 + 1)(\alpha_2 - \alpha_3 + 1) \ldots (\alpha_\rho - \alpha_{\rho+1})$$

whenever $p > 2$. In this case the total number of characteristic subgroups in

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* It may be observed that $G \equiv H_1$ whenever all the operators of order p^δ in G are conjugate under I for at least two values of $\delta > 0$. That is, while all the operators of highest order in G are always conjugate under I, all those of any other order are never conjugate unless all the invariants of G are equal.

G is given by the simple formula

$$\sum_{\rho=1}^{\rho=\lambda}(\alpha_1-\alpha_2+1)(\alpha_2-\alpha_3+1)\ldots\ldots(\alpha_{\rho-1}-\alpha_\rho+1)(\alpha_\rho-\alpha_{\rho+1}),\quad \alpha_{\lambda+1}=0.$$

This formula remains true when $p=2$ if there is no more than one cyclic $H_l\,(l \leqq \rho)$ for which both of the conditions

$$\alpha'_\rho > \alpha'_{\rho+1},\quad \alpha_{\rho-1}-\alpha'_{\rho-1} > \alpha_\rho-\alpha'_\rho$$

are satisfied. If there are $k>1$ such cyclic H_l's, there are as many additional C's which correspond to the conjugates of such a t as there are combinations of k things taken 2, 3, ..., k at a time. None of these is generated by the conjugates of t. It should be observed that *the formula*

$$\sum_{\rho=1}^{\rho=\lambda}(\alpha_1-\alpha_2+1)(\alpha_2-\alpha_3+1)\ldots\ldots(\alpha_{\rho-1}-\alpha_\rho+1)(\alpha_\rho-\alpha_{\rho+1})$$

gives the number of sets of operators in G which are conjugate under I regardless of the value of p. The identity is not included in this enumeration of sets of conjugate operators.

Among the special results, it may be well to observe that every characteristic subgroup which contains an operator of order p^δ which could be used as an independent generator of G must also contain all the operators of G whose orders divide p^δ. If a characteristic subgroup contains an operator of order p^δ that is found in H_ϵ, it contains also all the operators of $\{H_1, H_2, \ldots\ldots, H_\epsilon\}$ whose orders divide p^δ. If G contains one operator of order p^δ which could be used as an independent generator at least $\frac{p-1}{p}$ of its operators of order p^δ could be used separately as independent generators. In any abelian group of order p^α the number of operators of a given order is at least $p-1$ times the number of all those of lower orders. If a subgroup includes more than one-half of the operators of a given order, it must include all the operators of this order and of all lower orders. All the characteristic subgroups, which contain operators of order p^δ, have in common a characteristic subgroup which includes operators of order p^δ. The cyclic group is the only one in which every subgroup is characteristic and the groups of type (1, 1, 1, ...) is the only one which has no characteristic subgroup.

If the order of G is not a power of a single prime, the number of its sets of conjugate operators can be more readily stated if the identity is regarded as forming a conjugate set by itself. When this is done, the number of sets of conjugate operators in G is merely the product of the numbers of sets of conjugates in its Sylow subgroups. This product will also be the number of the characteristic subgroups of G (the identity being regarded as a characteristic subgroup) except when the special conditions noted above are satisfied in the Sylow subgroup of order 2^{a_0} contained in G.